

# Singular perturbation of nonlinear systems with regular singularity

William R. P. Conti<sup>\*</sup> and Domingos H. U. Marchetti<sup>†</sup>

## Abstract

We extend Balser-Kostov method of studying summability properties of a singularly perturbed inhomogeneous linear system with regular singularity at origin to nonlinear systems of the form

$$\varepsilon z f' = F(\varepsilon, z, f)$$

with  $F$  a  $\mathbb{C}^\nu$ -valued function, holomorphic in a polydisc  $\bar{D}_\rho \times \bar{D}_\rho \times \bar{D}_\rho^\nu$ . We show that its unique formal solution in power series of  $\varepsilon$ , whose coefficients are holomorphic functions of  $z$ , is 1-summable under a Siegel-type condition on the eigenvalues of  $F_f(0, 0, 0)$ . The estimates employed resemble the ones used in KAM theorem. A simple Lemma is developed to tame convolutions that appears in the power series expansion of nonlinear equations.

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## 1 Introduction

Balser and Kostov[BK] have studied singularly perturbed linear system with regular singularity at  $z = 0$  of the form

$$\varepsilon z f' = Af - b \tag{1.1}$$

$f'$  means derivative of  $f$  w.r.t.  $z$ ;  $A = A(\varepsilon, z)$  and  $b = b(\varepsilon, z)$  are, respectively, a  $\nu \times \nu$  matrix and a  $\nu$ -vector whose entries are holomorphic in the polydisc  $D_R \times D_R$ ,  $R > 0$ .<sup>1</sup>

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<sup>\*</sup>Address: Instituto de Física, Universidade de São Paulo, Caixa Postal 66318, 05314-970 São Paulo, SP, Brasil. Supported by FAPESP under grant #07/59739-4. E-mail: [wrpconti@if.usp.br](mailto:wrpconti@if.usp.br)

<sup>†</sup>Present Address: Mathematics Department, The University of British Columbia, Vancouver, BC, Canada V6T 1Z2. Email: [marchett@math.ubc.ca](mailto:marchett@math.ubc.ca)

<sup>1</sup>Here,  $D_\rho(z_0) = \{z \in \mathbb{C} : |z - z_0| < \rho\}$  denotes an open disc of radius  $\rho > 0$ , centered at  $z_0$ ,  $\bar{D}_\rho(z_0)$  denotes its closure and  $D_\rho = D_\rho(0)$ .

$A$  is, in addition, such that  $A(0,0)^{-1}$  exists. For such a system, there exists a unique formal solution in the ring  $\mathcal{O}(r)[[\varepsilon]]_1$  of formal power series

$$\hat{f}(\varepsilon, z) = \sum_{i=0}^{\infty} a_i(z) \varepsilon^i \quad (1.2)$$

in  $\varepsilon$  with coefficients  $a_i(z)$  in the ring  $\mathcal{O}(r)$  of holomorphic functions on  $D_r$ , continuous in its closure, satisfying

$$\max_{|z-z_0| \leq r} |a_i(z)| \leq C \mu^i i! , \quad i = 0, 1, 2, \dots \quad (1.3)$$

for some positive constants  $C, \mu$  and  $0 < r < R$ . The authors have shown (see Theorem 1 and 2 of [BK]) that  $\hat{f}(\varepsilon, z)$  is the 1–Gevrey asymptotic expansion of a holomorphic function  $f(\varepsilon, z)$  in  $S(\theta, \gamma; E) \times D_r$ , as  $\varepsilon$  tends to 0, if the closed sector  $\bar{S}(\theta, \gamma; E)$  does not contain any ray on the direction of the eigenvalues  $\lambda_j$  of  $A(0, 0)$ :

$$|\arg \lambda_j - \theta| > \gamma/2 , \quad j = 1, \dots, n . \quad (1.4)$$

The formal series  $\hat{f}(\varepsilon, z)$  is thus 1–summable in the direction  $\theta$  provided the eigenvalues of  $A(0, 0)$  satisfy a Siegel–type condition, i.e. the  $\lambda_j$  satisfy (1.4) for some  $\gamma \geq \pi$ .

A nonlinear version of (1.1) appears as follows. Let  $f(\varepsilon, z)$  be the unique extension in  $S(0, \gamma; E) \times D_r$ , with  $\varepsilon = 2/N$ , of the meromorphic function

$$\phi_\varepsilon(z) = \frac{i}{2\sqrt{z}} \frac{J_{N/2}(i\sqrt{z}N)}{J_{N/2-1}(i\sqrt{z}N)} \quad (1.5)$$

where  $J_\kappa(x)$  is the Bessel function of order  $\kappa$ . This function is related with the Fourier–Stieltjes transform  $\hat{\sigma}^N(x)$  of a uniform measure  $\sigma^N$  on the  $N$ –dimensional sphere of radius  $\sqrt{N}$  and we refer to [MC] and [MCG] for the motivations for its study. The  $N$  dependence in the argument is chosen in such way that  $\phi_\varepsilon(z)$  attains, as  $\varepsilon$  goes to 0, a limit function

$$\phi_0(z) = \frac{-1}{1 + \sqrt{1 + 4z}} \quad (1.6)$$

(see Proposition 2.1 of [MCG]).  $\phi_\varepsilon$  satisfies an ordinary (Riccati) differential equation

$$\varepsilon z \phi'_\varepsilon + \phi_\varepsilon - 2z \phi_\varepsilon^2 + \frac{1}{2} = 0 \quad (1.7)$$

which, despite of being nonlinear, can be dealt by Balser–Kostov’s method. It has been shown by the present authors (see Lemmas 3.2, 3.3 and 3.4 of [MC]) **(a)** existence of a unique formal solution  $\hat{\phi}_\varepsilon(z)$  in the form of (1.2), satisfying (1.3); **(b)**  $\hat{\phi}_\varepsilon(z)$  is the 1–Gevrey asymptotic expansion of the holomorphic solution  $f(\varepsilon, z)$  of (1.7) in  $S(0, \gamma; E) \times D_r$ , as  $\varepsilon$  goes to 0 in  $S(0, \gamma; E)$ ; **(c)** choosing the sector  $S(\theta, \gamma; E)$  of opening angle  $\gamma > \pi$  away from the negative real axis,  $\hat{\phi}_\varepsilon(z)$  is, in addition, 1–summable in  $\theta$  direction and its sum is equal to  $f(\varepsilon, z)$ .

In the present article statements **(a)**–**(c)**, together with the 1-summability, will be extended for more general ordinary differential equations of the form

$$\varepsilon z f' = F(\varepsilon, z, f) , \quad (1.8)$$

with  $f = (f^1, \dots, f^\nu)$  and  $F = (F^1, \dots, F^\nu)$   $\nu$ -vector functions,  $F^i$  holomorphic in a polydisc, say  $\bar{D}_\rho \times \bar{D}_{\rho_1} \times \bar{D}_\rho^\nu$ , for some  $\rho_1 > \rho > 0$ . As in ([BK]), the  $\nu \times \nu$  matrix  $A_{0,1}(0) = F_f(0, 0, 0)$  is assumed to be invertible, a condition that makes (1.8) to possess a regular singularity at  $z = 0$ , and every eigenvalue of  $A_{0,1}(0)$  satisfies condition (1.4). Equation (1.7) is of the form (1.8) with  $\nu = 1$  and<sup>2</sup>

$$F(\varepsilon, z, f) = -\frac{\beta(\varepsilon)}{2} - f + 2zf^2 \quad (1.9)$$

Balser–Kostov summability proof in [BK] of the formal series  $\hat{f}$  solution does not follow the usual route: the (formal) Borel transform  $\hat{\mathcal{B}}\hat{f}$  of  $\hat{f}$  is analytically continued along some sector of infinite radius (see e.g. [Ba]). Their proof establishes instead Grevrey asymptotic expansion directly from the equation (1.1), making resource of an auxiliary Lemma regarding an infinite system of linear equation of the same type. Although (1.8) is nonlinear, the system of infinitely many equations obtained by taking derivatives of (1.8) with respect to  $\varepsilon$  is linear, indeed of the type stated in Lemma 3 of [BK], and Balser–Kostov’s method carries over to equation of the form (1.8).

The layout of this paper is as follows. In Section 2 we prove existence of a unique solution of (1.8) in power series of  $z$ . In Section 3 we show that the formal power series solution of (1.8) is Gevrey of order 1. In Section 4 Gevrey asymptotics are established. Our main result, the 1-summability of the formal solution of (1.8), is stated in Section 5 and proved using Propositions 2.2-4.1 of the previous sections. The main ingredient (Lemma 2.3), is employed to tame arbitrarily large number of convolutions arised in the expansion of  $F$  in power series of  $f$ .

## 2 Power series in $z$

Under the hypothesis on  $F$ , the series

$$F(\varepsilon, z, f) = \sum_{\substack{n,m=0: \\ n+m \neq 0}}^{\infty} A_{n,m}(\varepsilon) z^n f^m \quad (2.1)$$

converges (in norm) absolutely in  $\bar{D}_{\rho_1} \times \bar{D}_\rho^\nu$ , uniformly in  $\varepsilon \in \bar{D}_\rho$ , with the coefficients

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<sup>2</sup>Statements **(a)**–**(c)** hold with  $1/2$  in (1.7) replaced by  $\beta(\varepsilon)/2$  for any 1-summable  $\beta(\varepsilon) = \sum_{n \geq 0} \beta_n \varepsilon^n$  formal series in  $\theta$  direction. In this case the limit function (1.6) is replaced by  $\phi_0 = -\beta_0/(1 + \sqrt{1 + 4\beta_0 z})$ .

$A_{n,m}(\varepsilon)$  regarded as a multilinear operator,

$$f^m \in \underbrace{\mathbb{C}^\nu \times \cdots \times \mathbb{C}^\nu}_{m \text{ copies}} \mapsto A_{n,m}(\varepsilon) f^m \in \mathbb{C}^\nu$$

$$(A_{n,m}(\varepsilon) f^m)^i = \sum_{i_1, \dots, i_m=1}^{\nu} A_{n,m}^{i, i_1, \dots, i_m}(\varepsilon) f^{i_1} \cdots f^{i_m}, \quad (2.2)$$

endowed with an operator norm induced by the Euclidean space  $\mathbb{C}^\nu$ :

$$\|A_{n,m}(\varepsilon)\| = \sup_{(v_1, \dots, v_m) \in \mathbb{C}^{m\nu}} \frac{\|A_{n,m}(\varepsilon) v_1 \cdots v_m\|}{\|v_1\| \cdots \|v_m\|},$$

holomorphic in  $D_\rho$  as a function of  $\varepsilon$ .

In (2.2) and from now on,  $f = (f^1, \dots, f^\nu)$  denotes a  $\nu$ -vector with  $i$ -th component  $f^i$  and Euclidean norm  $\|f\|^2 = f \cdot f = \sum_{i=1}^{\nu} \bar{f}^i f^i$ . Without loss of generality, we assume  $A_{00}(\varepsilon) \equiv 0$  and since the l.h.s. of (1.8) vanishes for  $z = 0$ , its solution in power series reads

$$f(\varepsilon, z) = \sum_{k=1}^{\infty} f_k(\varepsilon) z^k. \quad (2.3)$$

(by hypothesis  $f(\varepsilon, 0) \equiv 0$ ). For  $F$  given by the example (1.9),  $A_{0,0}(\varepsilon) = \beta(\varepsilon)/2$  does not vanishes and we may replace  $f$  and (1.9) by  $\tilde{f} = f + \beta/2$  and

$$\tilde{F} = (1 - 2\beta z) \tilde{f} + \frac{\beta^2}{2} z + 2z \tilde{f}^2$$

which satisfy  $\tilde{f}(\varepsilon, 0) = 0$  and  $\tilde{A}_{0,0}(\varepsilon) = 0$ . The general case differs very little from this particular example.

Substituting the power series (2.3) into (2.1) together with (1.8), we are led to a system of equations

$$(\varepsilon j I - A_{0,1}(\varepsilon)) f_j = g_j(\varepsilon; f_1, \dots, f_{j-1}) \quad (2.4)$$

with  $g_j = (g_j^1, \dots, g_j^\nu)$  given by

$$g_1^i(\varepsilon) = A_{1,0}^i(\varepsilon)$$

and

$$g_j^i(\varepsilon; f_1, \dots, f_{j-1}) = \sum_{\substack{n, m: \\ 2 \leq n+m \leq j}} \sum_{i_1, \dots, i_m=1}^{\nu} A_{n,m}^{i, i_1, \dots, i_m}(\varepsilon) (f^{i_1} * \cdots * f^{i_m})_{j-n} \quad (2.5)$$

for  $j \geq 2$ ; for any two sequences  $\alpha = (\alpha_k)_{k \geq 1}$  and  $\beta = (\beta_k)_{k \geq 1}$ , their convolution product  $\alpha * \beta = ((\alpha * \beta)_k)_{k \geq 1}$  is a sequence defined by  $(\alpha * \beta)_1 = 0$  and

$$(\alpha * \beta)_k = \sum_{l=1}^{k-1} \alpha_l \beta_{k-l}, \quad k \geq 2. \quad (2.6)$$

The restriction  $n + m \leq j$  in (2.5) results from the fact that our sequence  $f^i = (f_k^i)_{k \geq 1}$  starts with  $k = 1$  and a convolution involving  $m$  sequences cannot have nonvanishing component  $j - n$  if  $j > n + m$ .

Consequently, for any  $k \in \mathbb{N}$  arbitrary, (2.4) for  $1 \leq j \leq k$  forms a **closed** system of  $\nu \cdot k$  equations, involving  $\nu \cdot k$  unknown functions which can be solved by iteration starting from

$$f_1(\varepsilon) = (\varepsilon I - A_{0,1}(\varepsilon))^{-1} A_{1,0}(\varepsilon) . \quad (2.7)$$

If equation (2.4) for  $1 \leq j \leq k - 1$  and  $k \geq 2$  have been solved, then

$$f_k(\varepsilon) = (\varepsilon k I - A_{0,1}(\varepsilon))^{-1} g_k(\varepsilon; f_1, \dots, f_{k-1}) . \quad (2.8)$$

Regarding the inverse matrix  $(\varepsilon k I - A_{0,1}(\varepsilon))^{-1}$ , we have the following

**Lemma 2.1 (Lemma 1 of [BK])** *Suppose (1.4) holds with  $\theta = 0$  and  $\lambda_j$ ,  $j = 1, \dots, \nu$ , eigenvalues of  $A_{0,1}(0) = F_f(0, 0, 0)$ . One can always find  $E > 0$  such that, if  $k |\varepsilon| \geq \frac{c}{c-1} \sup_{|\varepsilon| \leq E} \|A_{0,1}(\varepsilon)\|$  for some  $c > 1$ , the inverse matrix in (2.8), given by*

$$(\varepsilon k I - A_{0,1}(\varepsilon))^{-1} = \sum_{n=0}^{\infty} \frac{1}{(\varepsilon k)^{n+1}} (A_{0,1}(\varepsilon))^n$$

*is bounded and satisfies  $\|(\varepsilon k I - A_{0,1}(\varepsilon))^{-1}\| \leq c$ , uniformly in  $S(0, \gamma; E)$ . If  $k |\varepsilon| < \frac{c}{c-1} \sup_{|\varepsilon| \leq E} \|A_{0,1}(\varepsilon)\|$ , let  $\lambda_j(\varepsilon)$ ,  $j = 1, \dots, n$ , the eigenvalues of  $A_{0,1}(\varepsilon)$ , be so that their distances from every ray  $\eta = r e^{i\tau}$  intercepting  $S(0, \gamma; E)$  are bounded from below by a constant  $a > 0$ :*

$$a = \inf \left\{ |\lambda_j(\varepsilon) - r e^{i\tau}| : 0 \leq r < \infty, |\tau| \leq \gamma, j = 1, \dots, n \text{ and } \varepsilon \in S(0, \gamma; E) \right\} . \quad (2.9)$$

Then,

$$|\det(\varepsilon k I - A_{0,1}(\varepsilon))| = \prod_{j=1}^{\nu} |\varepsilon k - \lambda_j(\varepsilon)| \geq a^{\nu} > 0$$

*together with the formula  $A^{-1} = \text{Adj}(A) / \det A$  for inverse of a matrix  $A$ , where  $\text{Adj}(A)$  is the transposed of the cofactors matrix of  $A$ , (see e.g. [La]) and with the boundedness in  $S(0, \gamma; E)$  of all cofactors of  $A_{0,1}(\varepsilon)$ , give*

$$\|(\varepsilon k I - A_{0,1}(\varepsilon))^{-1}\| \leq c , \quad (2.10)$$

*uniformly in  $S(0, \gamma; E)$  for every  $k \in \mathbb{N}$ .*

**Proposition 2.2** *Let  $F$  be given by (2.1) with the eigenvalues of  $A_{0,1}(0)$  obeying hypothesis (1.4). There exist  $\gamma$ ,  $E$  and  $\sigma$  such that (1.8) has a solution  $f(\varepsilon, z)$  holomorphic in  $S(0, \gamma; E) \times D_{\sigma}$ . The solution  $f(\varepsilon, z)$  converges, as  $\varepsilon \rightarrow 0$  in the sector  $S(0, \gamma; E)$ , to the unique solution  $f^*(z)$  of  $F(0, z, f) = 0$  in  $D_{\sigma}$  satisfying  $f(0) = 0$ .*

**Proof** Since (2.3) solves (1.8), its coefficients  $f_k(\varepsilon)$  satisfy the formal relations (2.4) whose solution depends on the existence of inverse matrix  $(\varepsilon k I - A_{0,1}(\varepsilon))^{-1}$  for every  $k \in \mathbb{N}$  and  $\varepsilon \in S(0, \gamma, E)$ . Assuming (1.4) holds for every eigenvalues of  $A_{0,1}(0)$ , let  $\gamma$  and  $E$  be such that (2.9), and consequently (2.10), holds. Hence,  $f_k(\varepsilon)$  given by (2.8) is bounded uniformly in  $S(0, \gamma; E)$ , uniquely defined for every  $k \in \mathbb{N}$  and, in view of these, holomorphic in  $S(0, \gamma; E)$ .

Let  $\phi_l$  and  $\alpha_{n,m}$  be the supremum in  $S(0, \gamma; E)$  of  $\|f_l(\varepsilon)\|$  and  $\|A_{n,m}(\varepsilon)\|$ , respectively:

$$\begin{aligned}\phi_l &= \sup_{\varepsilon \in S(0, \gamma; E)} \|f_l(\varepsilon)\| \\ \alpha_{n,m} &= \sup_{\varepsilon \in S(0, \gamma; E)} \|A_{n,m}(\varepsilon)\| .\end{aligned}\tag{2.11}$$

By Cauchy formula

$$\frac{1}{n!m!} F^{(0,n,m)}(\varepsilon, 0, 0) \left( \frac{f}{\|f\|} \right)^m = \frac{1}{(2\pi i)^2} \oint \oint \frac{F(\varepsilon, \zeta, \phi f / \|f\|)}{\zeta^{n+1} \phi^{m+1}} d\zeta d\phi$$

and there exists  $C < \infty$  ( $= \sup_{\bar{S}(0, \gamma; E) \times \bar{D}_{\rho_1} \times \bar{D}_\rho^\nu} \|F(\varepsilon, z, f)\|$ ,  $E \leq \rho$ ) such that

$$\alpha_{n,m} \leq \frac{C}{\rho_1^n \rho^m} .\tag{2.12}$$

Now, we prove that the majorant series  $\sum_{l=1}^{\infty} \phi_l \sigma^l$  converges and is bounded by  $\rho$  for some  $0 < \sigma < \rho$ . For this, the following lemma will be stated more generally than it is needed for this section.

**Lemma 2.3** *Let  $\lambda \geq 0$  be given and let  $A = (1 + \pi^2/3)^{-1}/2 = 0.1165536 \dots$ . Consider the sequence  $(C_l)_{l=0}^{\infty}$  with  $C_0 = A$  or  $C_0 = 0$  and <sup>3</sup>*

$$C_l = \frac{A l!^\lambda}{l^2} , \quad \forall l \geq 1 .$$

Then

$$\sum_{l=0}^m C_l C_{m-l} \leq C_m\tag{2.13}$$

holds for every  $m \geq 0$ .

**Proof** Since  $\binom{m}{l} \geq 1$ ,

$$\frac{1}{l} + \frac{1}{m-l} = \frac{m}{l(m-l)}$$

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<sup>3</sup>The sequence  $(C_l)$  of this Section has  $C_0 = 0$ . Lemma 2.3 has been stated with  $C_0 = A$  to be used elsewhere in another Section.

and  $0 \leq (a - b)^2 = 2(a^2 + b^2) - (a + b)^2$  holds for any real numbers  $a$  and  $b$ , we have

$$\begin{aligned} \frac{1}{C_m} \sum_{l=0}^m C_l C_{m-l} &\leq A \left( 2 + \sum_{l=1}^{m-1} \frac{m^2}{l^2(m-l)^2} \right) \\ &\leq 2A \left( 1 + \sum_{l=1}^{m-1} \left( \frac{1}{l^2} + \frac{1}{(m-l)^2} \right) \right) \\ &\leq 2A \left( 1 + \frac{\pi^2}{3} \right) = 1. \end{aligned}$$

□

It thus follows from (2.13) with  $C_0 = 0$  that

$$\sum_{\substack{l_1, \dots, l_k \geq 1: \\ l_1 + \dots + l_k = m}} C_{l_1} \cdots C_{l_k} \leq C_m \quad (2.14)$$

holds for any  $1 \leq k \leq m$ .

Let us assume that (2.12) can be written as (see Remark 3.1)

$$\alpha_{n,m} \leq \frac{\alpha}{c} C_n \frac{1}{\rho^{n+m}} \quad (2.15)$$

and suppose

$$\phi_l \leq \alpha C_l \frac{1}{\kappa^l}$$

holds for  $l \geq 1$ , with  $(C_l)_{l \geq 1}$  the sequence in Lemma 2.3 with  $\lambda = 0$ , for some  $\alpha$  and  $\kappa < \rho$ . Hence, by (2.7) together with (2.10) and (2.11), we have

$$\phi_1 \leq c \alpha_{1,0} \leq \alpha A \frac{1}{\kappa} \quad (2.16)$$

and, by (2.8) and (2.5) together with (2.10),

$$\|f_k(\varepsilon)\| \leq c \sum_{\substack{n,m: \\ 2 \leq n+m \leq k}} \|A_{n,m}(\varepsilon)\| (\|f(\varepsilon)\| * \cdots * \|f(\varepsilon)\|)_{k-n}$$

with  $\|f(\varepsilon)\|$  denoting the sequence  $(\|f_j(\varepsilon)\|)_{j \geq 1}$ . Taking the supremum over  $\varepsilon \in S(0, \gamma; E)$  in both sides together with (2.11), (2.12) and (2.14),

$$\begin{aligned} \phi_k &\leq c \left( \alpha_{k,0} + \sum_{\substack{n,m \geq 1: \\ 2 \leq n+m \leq k}} \alpha_{n,m} \left( \underbrace{\phi * \cdots * \phi}_m \right)_{k-n} \right) \\ &\leq \alpha C_k \frac{1}{\rho^k} + \alpha \frac{1}{\kappa^k} \sum_{1 \leq n \leq k-1} C_n \left( \frac{\kappa}{\rho} \right)^n C_{k-n} \sum_{1 \leq m \leq k-n} \left( \frac{\alpha}{\rho} \right)^m \\ &\leq \alpha \left( \frac{\kappa^k}{\rho^k} + \frac{\alpha}{\rho - \alpha} \right) C_k \frac{1}{\kappa^k} \leq \alpha C_k \frac{1}{\kappa^k}, \end{aligned}$$

holds for  $k \geq 2$  provided  $\alpha < \rho/2$  and

$$\kappa = \rho \sqrt{1 - \frac{\alpha}{\rho - \alpha}} \quad (2.17)$$

With  $\alpha$  and  $\kappa$  satisfying these conditions, we conclude

$$\phi_l = \sup_{\varepsilon \in S(0, \gamma; E)} \|f_l(\varepsilon)\| \leq \alpha \frac{A}{l^2} \frac{1}{\kappa^l}, \quad \forall l \geq 1 \quad (2.18)$$

and  $(f_l(\varepsilon) z^l)_{l \geq 1}$  is a sequence of holomorphic functions, uniformly bounded in  $S(0, \gamma; E) \times D_\sigma$  by  $(\phi_l \sigma^l)_{l \geq 1}$ , whose sum  $f(\varepsilon, z) = \sum_{l=1}^{\infty} f_l(\varepsilon) z^l$  is bounded (in norm) by

$$\sum_{l=1}^{\infty} \phi_l \sigma^l = \frac{\alpha A \kappa}{\kappa - \sigma} = \rho \quad (2.19)$$

provided  $\sigma < \kappa$  satisfies  $\sigma = \kappa(\rho - \alpha A)/\rho = (\rho - \alpha A)\sqrt{1 - \alpha/(\rho - \alpha)}$ , by (2.17). Under this choice of  $\sigma$ ,  $F(\varepsilon, z, D_\rho^\nu) \subset D_\sigma^\nu$  uniformly in  $S(0, \gamma; E) \times D_\sigma$  and the solution we have obtained by the formal expansion (2.4) and (2.5) acquires sense. The power series solution (2.3) of (1.8) thus converges to a unique analytic function  $f(\varepsilon, z)$  in  $S(0, \gamma; E) \times D_\sigma$ . The proof of uniqueness will be omitted.

From the uniform convergence of (2.3) we conclude that, for any fixed  $z \in D_\sigma$ , the solution  $f(\varepsilon, z)$  tends to

$$f(0, z) = \lim_{S(0, \gamma; E) \ni \varepsilon \rightarrow 0} \sum_{l=0}^{\infty} f_l(\varepsilon) z^l = \sum_{l=0}^{\infty} \lim_{S(0, \gamma; E) \ni \varepsilon \rightarrow 0} f_l(\varepsilon) z^l = f^*(z)$$

where  $f^*(z)$  is the unique solution of equation

$$F(0, z, f) = 0$$

for  $f$ , by the analytic implicit function theorem (see e.g. Section 2.3 of [Be] or the next section, for an alternative solution). Note that the solution  $f^*(z)$  is regular at  $z = 0$  since, by (2.3), it must satisfy  $f(0) = 0$  and this concludes the proof of Proposition 2.2.  $\square$

### 3 Formal power series in $\varepsilon$

As in (2.1), the double series

$$F(\varepsilon, z, f) = \sum_{n, m} B_{n, m}(z) \varepsilon^n f^m \quad (3.1)$$



converges (in norm) absolutely in  $\bar{D}_\rho \times \bar{D}_\rho^\nu$ , uniformly in  $z \in \bar{D}_{\rho_1}$ , with the coefficients  $B_{n,m}(\varepsilon)$  regarded as a multilinear operator  $f^m \in \mathbb{C}^{m\nu} \mapsto B_{n,m}(z)f^m \in \mathbb{C}^\nu$

$$(B_{n,m}(z)f^m)^i = \sum_{i_1, \dots, i_m=1}^{\nu} B_{n,m}^{i, i_1, \dots, i_m}(z) f^{i_1} \dots f^{i_m} .$$

By consistency,  $B_{00}(0) = 0$  but  $B_{00}(z)$  may not be identically zero. Before we go through the power series in  $\varepsilon$ , we study the solution  $f^*(z)$  of

$$0 = \sum_{m=0}^{\infty} B_{0,m}(z) a_0^m(z) , \quad (3.2)$$

in power series of  $z$ :

$$a_0(z) = \sum_{j=1}^{\infty} a_{0,j} z^j . \quad (3.3)$$

Note that  $f^*(z) = a_0(z)$ , by Proposition 2.2, so  $a_0(0) = 0$ . Replacing (3.3) into (3.2), and taking into account

$$B_{0,m}(z) = \sum_{n=0}^{\infty} A_{n,m}(0) z^n$$

equation (3.2) can be written as (omitting the argument  $\varepsilon = 0$  of  $A_{n,m}(0)$ , for simplicity)

$$0 = A_{j,0} + \sum_{1 \leq m \leq j} (A_{\cdot, m} * a_0 * \dots * a_0)_j .$$

For  $j = 1$ ,

$$A_{1,0} + A_{0,1} a_{0,1} = 0 \implies a_{0,1} = -A_{0,1}^{-1} A_{1,0} .$$

Now, supposing  $a_{0,1}, \dots, a_{0,k-1}$  have already been determined, then

$$a_{0,k} = -A_{0,1}^{-1} \left( A_{k,0} + \sum_{j=1}^{k-1} A_{j,1} a_{0,k-j} + \sum_{2 \leq m \leq k} (A_{\cdot, m} * a_0 * \dots * a_0)_k \right) . \quad (3.4)$$

If one takes the norm of (3.4), together with  $\|A_{0,1}^{-1}\| \leq c$ , (2.12), (2.14) and (2.18), that holds also for  $\varepsilon = 0$ ,

$$\|a_{0,k}\| \leq cC \left( \frac{1}{\rho_1^k} + \sum_{j=1}^{k-1} \left( \frac{1}{\rho_1} \right)^j \sum_{1 \leq m \leq k-j} \left( \frac{1}{\rho} \right)^m (\|a_{0,\cdot}\| * \dots * \|a_{0,\cdot}\|)_{k-j} \right) \leq \alpha C_k \frac{1}{\kappa^k} \quad (3.5)$$

provided we fix  $\alpha$  and  $\kappa$  as in the previous section, which is consistent with the domain in which  $f(\varepsilon, z)$  is holomorphic. This shows that  $f^*(z)$  is holomorphic in  $D_\sigma^\nu$  and proves the existence of a unique solution of  $F(0, z, f) = 0$  in the same domain.

**Remark 3.1** Regarding the radius of convergence of the power series of  $f^*(z)$  one can estimate it a little better using the Cauchy majorant method as in Section 3.2 of [Be] (see also [BK], Section 1, for the linear equation). Multiplying (3.5) by  $z^k$ , summing over  $k$  and replacing the inequality by equality, yields

$$\phi(z) = cC \frac{z/\rho_1}{1 - z/\rho_1} \frac{1}{1 - \phi(z)/\rho}$$

for a majorant  $\phi(z)$  of  $f^*(z)$ , whose solution

$$\phi(z) = \frac{\rho}{2} \left( 1 - \sqrt{\frac{1 - z/\sigma_1}{1 - z/\rho_1}} \right)$$

is holomorphic in a disc  $D_{\sigma_1}$  with  $\sigma_1 = \rho_1 \rho / (\rho + 4cC) < \rho_1$ , proportional to  $\rho_1$ . In Section 2, we have chosen  $\rho_1$  so large that (2.12) can be written as (2.15) and the radius of convergence  $\sigma$ , obtained applying Lemma 2.3 to convolutions, is proportional to  $\rho$  instead (see expression after (2.19)). Despite of this loss, the method introduced there is undeniably practical, more adaptable to diverse situations and, for these reasons, we shall apply it here and in further sections.

**Proposition 3.2** Suppose the formal power series (1.2) satisfies equation (1.8), formally, with  $F = F(\varepsilon, z, f)$  obeying the hypotheses stated after (1.8). Then, the coefficients  $(a_i(z))_{i \geq 0}$  of (1.2) are analytic functions of  $z$  in the open disc  $D_\kappa$  and there exist positive constants  $C$  and  $\mu$  such that

$$\|a_i(z)\| \leq Ci! \mu^i \quad (3.6)$$

holds for all  $i \geq 0$  and  $z \in \bar{D}_\sigma$ , with  $\sigma < \kappa < \rho$ . In other words, the formal power series is of Gevrey order 1, i.e.,  $\hat{f}(\varepsilon, z) \in \mathcal{O}(\sigma)[[\varepsilon]]_1$ .

**Proof** Substituting the power series (1.2) into (3.1), we are thus led to a system of equations

$$0 = \sum_{m=0}^{\infty} B_{0,m}(z) a_0^m(z) ,$$

which has already been solved for  $a_0(z)$ , and for  $i \geq 1$

$$z a'_{i-1}(z) = \sum_{m=1}^{\infty} m B_{0,m}(z) a_0^{m-1}(z) a_i(z) + \sum_{n=1}^i \sum_{m=1}^{\infty} B_{n,m} \left( \underbrace{a(z) * \cdots * a(z)}_m \right)_{i-n} . \quad (3.7)$$

We observe that the sum over  $m$  has no limit as the sequence  $a(z) = (a_k(z))_{k \geq 0}$  starts from  $k = 0$  and the convolution product of any two sequences  $\alpha = (\alpha_k)_{k \geq 0}$  and  $\beta = (\beta_k)_{k \geq 0}$  is now defined by

$$(\alpha * \beta)_k = \sum_{l=0}^k \alpha_l \beta_{k-l} , \quad k \geq 0 . \quad (3.8)$$

To isolate  $a_i$ , the largest index term in (3.7), we have to show that the matrix (recall  $B_{0,1}(0) = A_{0,1}(0)$ )

$$\begin{aligned} T_0(z) &= B_{0,1}(z) + \sum_{m=2}^{\infty} m B_{0,m}(z) a_0^{m-1} \\ &= A_{0,1}(0) \left( I + A_{0,1}(0)^{-1} \left( B_{0,1}(z) - B_{0,1}(0) + \sum_{m=2}^{\infty} m B_{0,m}(z) a_0^{m-1} \right) \right) \end{aligned} \quad (3.9)$$

is invertible for every  $z \in D_\kappa$  for some  $\kappa \leq \rho$ . For this, we take  $\kappa$  so small that

$$c \sup_{z \in D_\kappa(0)} \left( \|B_{0,1}(z) - B_{0,1}(0)\| + \sum_{m=2}^{\infty} m \|B_{0,m}(z)\| \|a_0\|^{m-1} \right) \leq b < 1$$

and, consequently,  $\|T_0(z)^{-1}\| \leq c/(1-b)$  holds uniformly in  $D_\kappa(0)$ .

It follows from (3.7) and (3.9) that

$$a_i(z) = T_0(z)^{-1} \left( z a'_{i-1}(z) - \sum_{n=1}^i \sum_{m=1}^{\infty} B_{n,m} \left( \underbrace{a(z) * \cdots * a(z)}_m \right)_{i-n} \right) \quad (3.10)$$

and this relation determines uniquely  $a_i(z)$  in terms of earlier coefficients. Note that  $a_i(z)$  is holomorphic in  $D_\kappa$  and, by (3.5) and (2.19)

$$\sup_{z \in D_\kappa(0)} |a_0(z)| \leq \delta A, \quad (3.11)$$

by letting  $\kappa$  small enough, for any  $\delta > 0$ . Now, to obtain an estimate on the growth rate of  $|a_i(z)|$ , let  $\varphi_i$  denote the  $i$ -th Nagumo norm

$$\|a_i\|_i := \sup_{z \in D_\kappa(0)} (d_\kappa(z))^i \|a_i(z)\|, \quad \text{where } d_\kappa(z) = \kappa - |z|. \quad (3.12)$$

of  $a_i(z)$  and let  $\beta_{n,m}$  the supremum in  $D_\kappa$  of  $\|B_{n,m}(z)\|$ . The properties we shall use on Nagumo's norms is proved in ([BK]) and references therein and are summarized by

1.  $\|f + g\|_k \leq \|f\|_k + \|g\|_k$ ;
2.  $\|fg\|_{k+l} \leq \|f\|_k \|g\|_l$ ;
3.  $\|f'\|_{k+1} \leq e(k+1)\|f\|_k$ ;
4.  $\|f\|_k \leq \kappa \|f\|_{k-1}$ ,

for any two functions  $f$  and  $g$  holomorphic in  $D_\kappa$  and nonnegative integers  $k, l$ .

Let us assume that

$$\varphi_l \leq \delta C_l \frac{1}{\nu^l} \quad (3.13)$$

holds for  $l = 1, 2, \dots, i-1$  with  $C_l = Al!/l^2$ , for some positive constants  $\delta$  and  $\nu$  to be determined. Similarly to (2.12) and (2.15),

$$\beta_{n,m} = \|B\|_0 \leq \frac{C_1}{\rho_1^n \rho^m} \leq \frac{\delta(1-b)}{c} \frac{\delta C_n}{\rho^{n+m}} \quad (3.14)$$

holds for some  $C_1 < \infty$  and  $\rho_1$  large enough. Then, it follows by (3.10), (3.14), (2.14) and the properties of Nagumo norms

$$\begin{aligned} \varphi_i &\leq \frac{c}{1-b} \left( \|z\|_0 \|a'_{i-1}\|_i + \sum_{n=1}^i \sum_{m=1}^\infty \beta_{n,m} \kappa^n \sum_{\substack{i_1, \dots, i_m \geq 0: \\ i_1 + \dots + i_m = i-n}} \varphi_{i_1} \cdots \varphi_{i_m} \right) \\ &\leq \frac{c}{1-b} e \kappa i \varphi_{i-1} + \delta \frac{1}{\nu^i} \sum_{n=1}^i \sum_{m=1}^\infty C_{i-n} \left( \frac{\kappa}{\rho} \right)^n C_n \left( \frac{\delta}{\rho} \right)^m \\ &\leq \left( \frac{2c}{1-b} e \kappa \nu + \frac{\delta}{\rho - \delta} \right) \delta C_i \frac{1}{\nu^i} \leq \delta C_i \frac{1}{\nu^i}, \end{aligned} \quad (3.15)$$

where the last inequality holds provided  $\delta < \rho/2$  and

$$\nu \leq \frac{1-b}{2ce\kappa} \left( 1 - \frac{\delta}{\rho - \delta} \right) \quad (3.16)$$

and this completes the induction:

$$\sup_{z \in D_\kappa(0)} (d_\kappa(z))^l |a_l(z)| \equiv \|a_l\|_l \leq \delta \frac{Al!}{l^2} \frac{1}{\nu^l} \quad \forall l \geq 1. \quad (3.17)$$

with  $\delta$  and  $\nu$  fixed so that (3.11) and (3.16) hold.

By definition (3.12) of Nagumo norm,

$$\|a_i(z)\| \leq \frac{1}{(\kappa - \sigma)^i} \|a_i\|_i \leq Ci! \mu^i$$

holds for all  $i \geq 1$  uniformly in  $\bar{D}_\sigma(0)$  for some  $\sigma < \kappa$ , with  $C = \delta A$  and  $\mu^{-1} = \nu(\kappa - \sigma)$ , which concludes the proof of Proposition 3.2.  $\square$

## 4 Gevrey asymptotics

In order to set up an equation involving derivatives of  $f$  with respect to  $\varepsilon$ , we write

$$\begin{aligned}\phi_i(\varepsilon, z) &= \frac{1}{i!} \frac{\partial^i f}{\partial \varepsilon^i}(\varepsilon, z) \\ \phi'_i(\varepsilon, z) &= \frac{\partial \phi_i}{\partial z}(\varepsilon, z)\end{aligned}$$

and  $\phi(\varepsilon, z) = (\phi_i(\varepsilon, z))_{i \geq 0}$  for the sequence of those functions defined on  $S(0, \gamma; E) \times D_\kappa(0)$ ; analogously to (2.1) and (3.1), we write

$$\begin{aligned}F(\varepsilon, z, f) &= \sum_{m=0}^{\infty} C_m(\varepsilon, z) f^m \\ F^{[i, 0, 0]}(\varepsilon, z, f) &= \sum_{m=0}^{\infty} C_m^{[i, 0]}(\varepsilon, z) f^m\end{aligned}$$

for the  $i$ -th derivative of  $h$  with respect to the first argument divided by  $i!$ . The  $i$ -th total derivative of  $F$  with respect to  $\varepsilon$  can thus be written as

$$\begin{aligned}G_i(\varepsilon, z, \phi_0, \dots, \phi_i) &= \frac{1}{i!} \frac{d^i}{d\varepsilon^i} F(\varepsilon, z, f) \\ &= \sum_m \left( C_m^{[i, 0]}(\varepsilon, z) * \underbrace{\phi(\varepsilon, z) * \dots * \phi(\varepsilon, z)}_m \right)_i \\ &= T(\varepsilon, z) \phi_i + \tilde{G}_i(\varepsilon, z, \phi_0, \dots, \phi_{i-1})\end{aligned} \tag{4.1}$$

where

$$T(\varepsilon, z) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} m A_{n, m}(\varepsilon) z^n \phi_0(\varepsilon, z)^{m-1} \tag{4.2}$$

and  $\tilde{G}_i(\varepsilon, z, \phi_0, \dots, \phi_{i-1})$  depends only on derivatives of  $f$  with respect to  $\varepsilon$  of order lower than  $i$ .

Differentiating equation (1.8)  $i$  times with respect to  $\varepsilon$ , dividing by  $i!$ , we have

$$\varepsilon z \phi'_i - T(\varepsilon, z) \phi_i = H_i(\varepsilon, z) \tag{4.3}$$

for  $i \geq 1$ , where

$$H_i(\varepsilon, z) = \tilde{G}_i(\varepsilon, z, \phi_0, \dots, \phi_{i-1}) - z \phi'_{i-1}, \tag{4.4}$$

may be think as inhomogeneous holomorphic function of  $(\varepsilon, z)$  in  $S(0, \gamma; E) \times D_\sigma(0)$ , and for  $i = 0$  simply (1.8):

$$\varepsilon z \phi'_0 = F(\varepsilon, z, \phi_0).$$

**Proposition 4.1** *Let  $f(\varepsilon, z)$  be the unique holomorphic solution of (1.8) on  $S(0, \gamma; E) \times \bar{D}_\sigma(0)$  with  $\sigma, \gamma$  and  $E$  as in Proposition 2.2. There exist  $0 < \sigma_1 \leq \sigma$ ,  $0 < E_1 \leq E$  and positive constants  $C$  and  $\mu$  such that*

$$\|\phi_i(\varepsilon, z)\| \leq C i! \mu^i$$

*holds for all  $i \geq 0$  and every point  $(\varepsilon, z)$  in  $S(0, \gamma; E_1) \times \bar{D}_{\sigma_1}(0)$ .*

**Proof** The case  $i = 0$  follows straightforwardly from Proposition 2.2. (4.3) is a linear singular perturbation equation with regular singularity which can be dealt with the following auxiliary result due to Balser-Kostov [BK] (see Lemma 3 therein). For this, we drop temporarily all subindices  $i$  in (4.3).

Let

$$T(\varepsilon, z) - t_0(\varepsilon) = \sum_{n=1}^{\infty} t_n(\varepsilon) z^n = S(\varepsilon, z) \quad (4.5)$$

and consider a sequence  $(\psi_k(\varepsilon, z))_{k \geq 0}$  satisfying the system

$$\begin{cases} \varepsilon z \psi'_0(\varepsilon, z) - t_0(\varepsilon) \psi_0(\varepsilon, z) = H(\varepsilon, z) \\ \varepsilon z \psi'_k(\varepsilon, z) - t_0(\varepsilon) \psi_k(\varepsilon, z) = S(\varepsilon, z) \psi_{k-1}(\varepsilon, z), \quad k = 1, 2, \dots \end{cases} \quad (4.6)$$

By (4.5) and linearity, the sum over all equations in (4.6) yields an equation of the form (4.3) satisfying by the sum  $\psi(\varepsilon, z) = \sum_{k=0}^{\infty} \psi_k(\varepsilon, z)$ . We assume that  $H(\varepsilon, z)$  admits an expansion

$$H(\varepsilon, z) = \sum_{n=0}^{\infty} h_n(\varepsilon) z^n \quad (4.7)$$

absolutely convergent for  $|z| \leq \sigma$ , uniformly in  $S(0, \gamma; E)$ . For  $H$  given by (4.1) and (4.4) this will actually be proven by induction when we resume the proof of Proposition 4.1. We write, in addition,  $f(z) \ll F(z)$  if  $f(z) = \sum_{k=0}^{\infty} c_k z^k$  is majorized by  $F(z) = \sum_{k=0}^{\infty} C_k z^k$ , i. e., if  $|c_k| \leq C_k$  holds for all  $k$ . If  $f$  is a  $\nu$ -vector or a  $\nu \times \nu$  matrix  $f(z) \ll F(z)$  means majorized relation for each component.

**Lemma 4.2** *There exist unique functions  $(\psi_k(\varepsilon, z))_{k \geq 0}$ , holomorphic in  $S(0, \gamma; E_1) \times \bar{D}_\sigma(0)$ , satisfying (4.6). Each  $\psi_k(\varepsilon, z)$  has a zero of order  $k$  at  $z = 0$ :  $\psi_k^{(0,k)}(\varepsilon, 0) = 0$ , and satisfies*

$$\psi(\varepsilon, z) = \sum_{k=0}^{\infty} \psi_k(\varepsilon, z) \ll \frac{a}{I - a\Gamma(z)} \Omega(z) \quad (4.8)$$

where

$$\Omega(z) = \sum_{n=0}^{\infty} \sup_{\varepsilon \in S(0, \gamma; E)} |h_n(\varepsilon)| z^n \quad (4.9)$$

$$\Gamma(z) = \sum_{n=0}^{\infty} \sup_{\varepsilon \in S(0, \gamma; E)} |t_n(\varepsilon)| z^n. \quad (4.10)$$

holds for some  $a < \infty$  provided  $\sigma_1$  is small enough.  $\psi(\varepsilon, z)$  is, in addition, the unique analytic solution in  $S(0, \gamma; E_1) \times D_\sigma(0)$  of

$$\varepsilon z \psi'(\varepsilon, z) - T(\varepsilon, z) \psi(\varepsilon, z) = H(\varepsilon, z) \quad (4.11)$$

with  $\psi(\varepsilon, 0) = 0$ .

**Proof** Plugging

$$\psi_k(\varepsilon, z) = \sum_{n=k}^{\infty} \varpi_{n,k}(\varepsilon) z^n$$

into (4.6), yields

$$\begin{aligned} (\varepsilon n I + t_0(\varepsilon)) \varpi_{n,0}(\varepsilon) &= h_n(\varepsilon), \quad n \geq 0 \\ (\varepsilon n I + t_0(\varepsilon)) \varpi_{n,k}(\varepsilon) &= \sum_{m=k-1}^{n-1} t_{n-m}(\varepsilon) \varpi_{m,k-1}(\varepsilon), \end{aligned}$$

for  $1 \leq k \leq n$  and  $n \geq 1$ . Observe that, by (4.2) and (4.5), together with the fact that  $\phi_0(\varepsilon, 0) = \sum_{j \geq 1} a_j(0) \varepsilon^j$  (recall  $a_0(0) = 0$ ),

$$\varepsilon n I + t_0(\varepsilon) = (\varepsilon n I + A_{0,1}(\varepsilon)) \left( I + (\varepsilon n I + A_{0,1}(\varepsilon))^{-1} \sum_{m \geq 1} m A_{0,m}(\varepsilon) \phi_0(\varepsilon, 0)^{m-1} \right)$$

is invertible for every  $\varepsilon \in D_{E_1}(0)$  if we take  $E_1 \leq E$  so small that

$$c \sup_{\varepsilon \in \bar{D}_{E_1}(0)} \sum_{m=2}^{\infty} m \|A_{0,m}(\varepsilon)\| \|\phi_0(\varepsilon, 0)\|^{m-1} \leq d < 1$$

and  $\|(\varepsilon n I + t_0(\varepsilon))^{-1}\| \leq c/(1-d) \equiv a < \infty$  holds uniformly in  $D_{E_1}(0)$ .

From these relations, we have

$$\psi_0(\varepsilon, z) = \sum_{n=0}^{\infty} \frac{1}{\varepsilon n I + t_0(\varepsilon)} h_n(\varepsilon) z^n,$$

and

$$\begin{aligned} \psi_k(\varepsilon, z) &= \sum_{n=k}^{\infty} \frac{1}{\varepsilon n I + t_0(\varepsilon)} \sum_{m=k-1}^{n-1} t_{n-m}(\varepsilon) \varpi_{m,k-1}(\varepsilon) z^n \\ &= \sum_{m=k-1}^{n-1} \left( \sum_{l=1}^{\infty} \frac{1}{\varepsilon(m+l)I + t_0(\varepsilon)} t_l(\varepsilon) z^l \right) \varpi_{m,k-1}(\varepsilon) z^m. \end{aligned}$$

Defining

$$\Psi_k(z) = \sum_{n=k}^{\infty} \sup_{\varepsilon \in S(0, \gamma; E_1)} |\varpi_{n,k}(\varepsilon)| z^n,$$

it follows, by (2.10), (4.9) and (4.10) that

$$\begin{aligned} \Psi_0(|z|) &\leq a\Omega(|z|) \\ \Psi_k(|z|) &\leq a\Gamma(|z|)\Psi_{k-1}(|z|) \end{aligned}$$

for  $k \geq 1$ . Since  $\psi_k(\varepsilon, z) \ll \Psi_k(z)$  for  $k \geq 1$  and  $\psi_0(\varepsilon, z) \ll a\Omega(z)$  for  $k = 0$  hold for all  $(\varepsilon, z) \in S(0, \gamma; E_1) \times \bar{D}_\sigma(0)$ , we conclude (4.8) provided the geometric series  $\sum_{k \geq 1} a^k \|\Gamma(\sigma_1)\|^k$  converges. By (4.5) and (4.2)

$$\|\Gamma(\sigma_1)\| = \sum_{n=1}^{\infty} \sup_{\varepsilon \in S(0, \gamma; E_1)} \|t_{n-1}(\varepsilon)\| \sigma^n < \frac{1}{a}$$

if  $\sigma$  is small enough and thence,  $\sum_{k=0}^{\infty} \psi_k(\varepsilon, z) = \psi(\varepsilon, z)$  is a uniformly convergent series of analytic functions in  $S(0, \gamma; E_1) \times D_\sigma(0)$  which solves (4.11). Since no other solution of (4.11), regular at  $z = 0$ , exists, the proof of Lemma 4.2 is concluded.  $\square$

We continue the proof of Proposition 4.1. It remains to show that the series (4.7) is uniformly convergent in  $S(0, \gamma; E_1) \times D_\sigma(0)$ . This follows by induction. Clearly,  $h_0(\varepsilon, z)$  is holomorphic in  $S(0, \gamma; E_1) \times D_\sigma(0)$ . Suppose that  $\phi_j(\varepsilon, z)$  is holomorphic in  $S(0, \gamma; E_1) \times D_\sigma(0)$  for each  $1 \leq j < i$ . Then, by (4.4),  $h_i(\varepsilon, z)$ , is holomorphic in the same domain. By Lemma 4.2,  $\phi_i(\varepsilon, z)$  is holomorphic in  $S(0, \gamma; E_1) \times D_\sigma(0)$  and, by (4.4), we conclude it also holds for  $h_{i+1}(\varepsilon, z)$ , justifying its representation as a convergent series (4.7), uniformly in  $S(0, \gamma; E_1) \times D_\sigma(0)$ . By induction,  $\phi_i(\varepsilon, z)$  is holomorphic in  $S(0, \gamma; E_1) \times D_\sigma(0)$  for each  $i \geq 1$  and

$$\phi_i(\varepsilon, z) \ll \frac{a}{I - a\Gamma(z)} \Omega_i(z) \ll \frac{a}{I - a\Gamma(\sigma_1)} \Omega_i(z). \quad (4.12)$$

where  $\Gamma_i$  depends on the  $\phi_j(\varepsilon, z)$  with  $j < i$ . For  $i = 0$ , by (4.9),

$$|f(\varepsilon, z)| = |\phi_0(\varepsilon, z)| \leq \frac{a}{I - a\Gamma(\sigma_1)} \Omega_0(|z|) \leq e_0 \quad (4.13)$$

holds for all  $\varepsilon \in S(0, \gamma; E_1)$  and  $z \in D_\sigma(0)$ . For  $i \geq 1$ , we consider the modification of Nagumo norms:

$$\|f\|_j = \sup_{z \in D_\sigma(0)} (d_{\sigma_1}(z))^j \sum_{n=0}^{\infty} \sup_{\varepsilon \in S(0, \gamma; E)} \frac{1}{n!} \left\| \frac{\partial^n f}{\partial z^n}(\varepsilon, 0) \right\| |z|^n,$$



with  $d_\sigma(z) = \sigma - |z|$ . It follows from (4.12) that

$$\|\phi_i\|_i \leq \frac{a}{I - a\Gamma(\sigma_1)} \|H_i\|_i, \quad (4.14)$$

where, by (4.1),

$$\|H_1\|_1 \leq \sum_m \|C_m^{[1,0]}\|_1 \|\phi\|_0^m \leq C \frac{\|\phi\|_0}{\rho - \|\phi\|_0}$$

and, together with the properties of Nagumo norms, for  $i \geq 2$

$$\|H_i\|_i \leq \|z\|_0 \|\phi'_{i-1}\|_i + \sum_m \sum_{\substack{i_0, \dots, i_m \geq 0: \\ i_0 + \dots + i_m = 1}} \|C_m^{[i_0, 0]}\|_{i_0} \|\phi_{i_1}\|_{i_1} \cdots \|\phi_{i_m}\|_{i_m}. \quad (4.15)$$

From these, together with (4.14), a recursive relation of the same type studied in Section 3 may be derived for the  $\|\phi_l\|_l$  (see (3.13)-(3.17)) and one may conclude that<sup>4</sup>

$$\|\phi_l\|_{l-1} \leq \Delta \frac{A!}{l^2} \omega^l \quad (4.16)$$

holds for all  $l \geq 1$  and some suitable constants  $\Delta$  and  $\omega$ . Picking  $\sigma_1 < \sigma$  together with the property of Nagumo norms, yields

$$|\phi_i(\varepsilon, z)| \leq \frac{\sigma}{(\sigma - \sigma_1)^i} \|\phi_i\|_{i-1} \leq C i! \mu^i$$

for all  $i \geq 1$  uniformly in  $S(0, \gamma; E_1) \times \bar{D}_{\sigma_1}(0)$ , with  $C = \sigma \Delta A$  and  $\mu = \omega/(\sigma - \sigma_1)$ . We choose  $C = \max(e_0, \sigma_1 \Delta A)$  in order to include the  $i = 0$  case. This concludes the proof of Proposition 4.1.

□

## 5 Summability

**Theorem 5.1** *Let (1.8) be considered with  $F$  given by (2.1) where the eigenvalues of  $A_{0,1}(0)$  obey hypothesis (1.4) for  $(\varepsilon, z)$  in a domain  $S(0, \gamma; E) \times D_\sigma(0)$  with  $\gamma > \pi$ . Then, there exist a radius  $\sigma > 0$  such that for  $z \in \bar{D}_\sigma(0)$  the formal solution  $\hat{f}(\varepsilon, z)$  is 1-summable in  $\theta = 0$  direction.*

**Proof** By Taylor's Theorem

$$r_I(\varepsilon, z) = \varepsilon^{-I} \left( f(\varepsilon, z) - \sum_{i=0}^{I-1} f_i(z) \varepsilon^i \right) = \frac{I}{\varepsilon^I} \int_0^\varepsilon f_I(\zeta, z) (\varepsilon - \zeta)^{I-1} d\zeta,$$

---

<sup>4</sup>The details for this estimate are left to the reader.

where the integral is along a path from 0 to  $\varepsilon$  inside  $S(0, \gamma; E)$ . This, together with Proposition 4.1, implies

$$|r_I(\varepsilon, z)| \leq CI!^{s'} \mu^I$$

for every  $I$  and  $(\varepsilon, z) \in S' \times \bar{D}_\sigma(0)$ , with  $S'$  any proper subsector of  $S(0, \gamma; E)$ . In addition, Proposition 3.2 states that  $\hat{f}(\varepsilon, z)$ , a formal solution of (1.8), is an element of  $\mathcal{O}(\sigma)[[\varepsilon]]_1$ ; therefore is an element of  $\mathcal{O}(\sigma)[[\varepsilon]]_1$  for any  $\sigma_1 < \sigma$ . Take now  $\sigma_1$  and  $E_1$  sufficiently small. Hence, by definition (see Section 1.5 of [Ba]),  $\hat{f}(\varepsilon, z)$  is an asymptotic expansion of order 1, as  $\varepsilon \rightarrow 0$  in the sector  $S(0, \gamma; E_1)$ , of  $f(\varepsilon, z)$ , which by Proposition 2.2 is an analytic solution of (1.8) in the domain  $S(0, \gamma; E_1) \times \bar{D}_{\sigma_1}(0)$ . Then, as  $\gamma > \pi$ , by hypothesis,  $f(\varepsilon, z)$  is the only Gevrey order 1 asymptotic expandable function in  $S(0, \gamma; E_1)$  which has  $\hat{f}(\varepsilon, z)$  as its asymptotic expansion, and  $\hat{f}(\varepsilon, z)$  is 1-summable in  $\theta = 0$  direction (see e.g. Section 3.2 of [Ba]).

□

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